Vanishing evaluations of biset functors

Serge Bouc, Radu Stancu & Jacques Thévenaz

LAMFA, UPJV & CTG, EPFL

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Let $A \trianglelefteq B \leq G$ and $\sigma \in \mathrm{Iso}(H, G)$. Then

$$ {}_G G_B\,, \quad {}_B G_G\,, \quad {}_B\, B/A_{\,B/A}\,, \quad {}_{B/A}\, B/A_{\,B}\,, \quad {}_H G_G $$

are transitive bisets.

Notation

For the above examples,

$$ \mathrm{Ind}_B^G\,, \quad \mathrm{Res}_B^G\,, \quad \mathrm{Inf}_{B/A}^B\,, \quad \mathrm{Def}_{B/A}^B\,, \quad \mathrm{Iso}_\sigma\,. $$

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$B(G, H)$ is the Grothendieck group of isomorphism classes of (G, H)-bisets with $[X] + [Y] = [X \coprod Y]$.

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- G, H, K finite groups. External multiplication :

$$\begin{array}{ccccc} kB(G,H) & \times & kB(H,K) & \to & kB(G,K) \\ (\quad [X] & , & [Y] \quad) & \mapsto & [X\times_H Y] \end{array}$$

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- $kB(G,G)$ is the double Burnside ring.

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[Rognerud, '15] $kB(A_5, A_5)$ is of infinite global dimension.
(*for* $\mathrm{char}\, k \neq 2, 3, 5$)

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Objets : finite groups

Morphisms : $\mathrm{Hom}_{\mathcal{C}}(G, H) = \quad B(H, G)$.

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- $S \in \mathcal{F}$ is simple if $\forall T \in \mathcal{F}$ with $T \subseteq S$ we have $T = S$ or $T = 0$.

Simple biset functors

For H a finite group, $V \in kB(H,H)-\mathrm{Mod}$

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$$J_{H,V}(X) := \bigcap_{\varphi \in kB(H,X)} \mathrm{Ker}\,(\,L_{H,V}(\varphi) : L_{H,V}(X) \to L_{H,V}(H)\,)\ .$$

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- if V is simple then $J_{H,V}$ is the unique maximal subfunctor of $L_{H,V}$.
- $S_{H,V} = L_{H,V}/J_{H,V}$ is simple and all simple biset functors are obtained this way.

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Let $F \in \mathcal{F}$ *and* $S \in \mathcal{F}$ *simple, with* $S(G) \neq 0$. *Then* $S \sqsubseteq F$ *if and only if* $S(G) \sqsubseteq F(G)$.

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Proposition (Rognerud, '15)

If $S_{H,V}(G) \neq 0$ for all $H \sqsubseteq G$ then

$$kB(G,G) - Mod \simeq \langle S_{H,V} \mid H \sqsubseteq G \rangle \, .$$

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Proposition

$\varepsilon : kB(-,H) \to k\bar{B}(-,H)$ induces $\varepsilon : L_{H,V} \to \bar{L}_{H,V}$ and, if V is simple, $\bar{\varepsilon} : S_{H,V} = L_{H,V}/J_{H,V} \to \bar{L}_{H,V}/\varepsilon(J_{H,V})$ is an isomorphism.

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Lemma

$$k\bar{B}(G,H) = \bigoplus_{(T,S)\in[\Sigma_H(G)/G]} \mathrm{Indinf}_{T/S}^{G} \circ k\bar{B}(T/S,H)\,.$$

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(B, A) and (T, S) are linked if $(B \cap T, A \cap S) \preceq (B, A)$ and $(B \cap T, A \cap S) \preceq (T, S)$. Notation : $(B, A) - (T, S)$.

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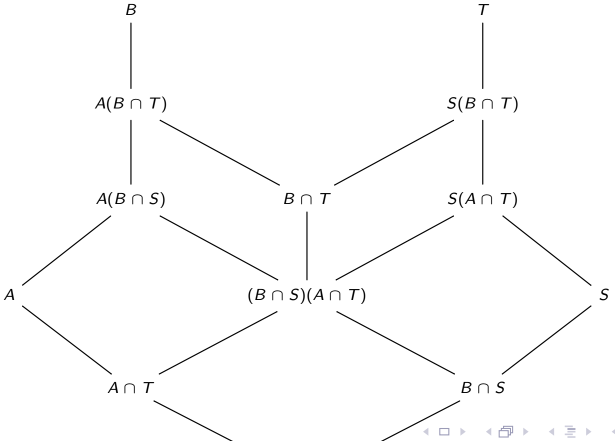

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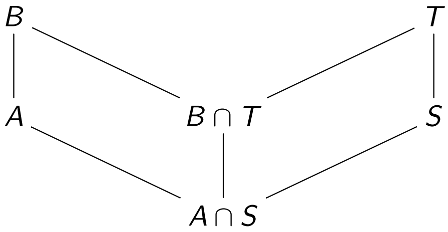

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In this case $\mathrm{Defres}^G_{T/S}\mathrm{Indinf}^G_{T/S} = \mathrm{id}_{T/S} + I(H,H)$.

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- $S_{H,V}(G) = 0$.
- *For any $(B,A),(T,S) \in \Sigma_H(G)$, the action on V of*

$$\sum_{\substack{g\in[B\backslash G/T]\\ (B,A)-{}^g(T,S)}} \sigma_{B/A}^{-1}\,\varphi_{B/A,{}^gT/{}^gS}\,\mathrm{Conj}_g\,\sigma_{T/S} \in kB(H,H)$$

is zero, where $\varphi_{B/A,{}^gT/{}^gS} : {}^gT/{}^gS \to B/A$ denotes the isomorphism induced by the linking $(B,A)-{}^g(T,S)$.

Minimal sections

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If $\Sigma_H(G) = \Sigma_H(G)^{\min}$*, then*

$$S_{H,V}(G)^{\min} \simeq \bigoplus_{(T,S)\in[\Sigma_H(G)/G]} \mathrm{Tr}_1^{\bar{N}_G(T,S)}(V) \ .$$